EXERCISE SHEET

1) Show that
$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(z)\frac{dz}{z}$$
for $\{z \in \mathbb{C} \,|\, |z| \leq 1\}$ and $k \geq 2$. Deduce that $\mathrm{Li}_k(z)$ can be analytically continued to a multi-valued function on $\mathbb{C} \setminus \{0,1\}$.

2) Let
$$D : \mathbb{C} \setminus \{0,1\} \to \mathbb{R}, \qquad z \mapsto \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$$
be the Bloch-Wigner function.
(a) Show that $D(\overline{z}) = -D(z)$ (in particular, D vanishes on $\mathbb{R}$).
(b) Show that $D(1-z) + D(z) = 0$.

3) Let $F = \mathbb{Q}(\sqrt{-a})$ be an imaginary quadratic field. Let d be the discriminant and let $\zeta_F(s)$ be the Dedekind zeta function of F. Show that $\zeta_F(2)$ can be written as
$$\zeta_F(2) = \frac{\pi^2}{6\sqrt{|d|}} \sum_{n=1}^{|d|-1} \left(\frac{d}{n}\right) D(e^{2\pi i n/|d|})$$
where $\left(\frac{d}{n}\right)$ is a periodic function of period $|d|$ taking values in $\{-1,0,1\}$ (for those who know: it is a Dirichlet character on $(\mathbb{Z}/|d|\mathbb{Z})^*$).

4) a) Compute $\zeta_{\mathbb{Q}(i)}(2)$.
b) Compute $\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$.

5) Show that
$$\xi := 2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right] \in \mathcal{B}(\mathbb{Q}(\sqrt{-7})\,.$$